Chapter 3.4
Upper Bounds on the number of resonances

Motivation

- if we have the operator $-\Delta + V$ on a bounded domain (eg T^N) then we have a discrete spectrum, and for $V \in L^\infty(T^n, R)$, **Weyl's law** gives us
- $|\{\lambda: \lambda^2 \in Spec(-\Delta_{T^n} + V), |\lambda| \leq r\}| = c_n vol(T^n) r^n \left(1 + O\left(\frac{1}{r}\right)\right)$
- with $c_n = \frac{2vol\left(B_{R^n}(0,1)\right)}{(2\pi)^n}$
- 1-d $\sum\{m_R(\lambda): |\lambda| \leq r\} = \frac{2\,|chsupp\, V|}{\pi} r\left(1 + o(1)\right)$

Theorem 3.27 (Upper bounds on the number of resonances)

- for $n \geq 3, odd$ and $V \in L^{\infty}_{comp}(R^n; C)$, let $m_R(\lambda)$ be multiplicity of resonance λ, then:
- $\sum_{|\lambda| \leq r} m_R(\lambda) \leq C_V r^n$
- history: r^{n+1} was proved by Melrose in '84, r^n proved by Zworski in '89, there is an upper bound proof for even dimensions (Vodev '94), lower bounds are still unknown

Idea

- Bound m_R by $m_h(\lambda)$ which is the number of zeros of $H(\lambda) = \det(I - (VR_0(\lambda)\rho)^{n+1})$. Then control the number of zeros using complex analysis by the growth of H.

Define $H(\lambda)$

- Let $H_V(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1})$ with $\rho \in C_c^\infty$ 1 on support of V
- To do this we need:
- **Lemma 3.24** (Trace class properties)
- $V, \rho \in L^\infty_{comp}(R^n, C)\ n \geq 3, odd$
- $(VR_0(\lambda)\rho)^p, p \geq \frac{n+1}{2}$
- is an entire family of trace class operators

Review of trace class operators

- For a compact operator $A: H_1 \to H_2$ on Hilbert spaces, we can write
- $A = \sum_0^\infty s_j (e_j \otimes f_j)$
- $s_0 \geq s_1 \geq \to 0$, e_j, f_j orthonormal systems. $s_j(A)$ are called **singular values**.
- An operator is trace class if $\sum s_j(A) < \infty$, so we must show:
- $\sum_j s_j\big((VR_0(\lambda)\rho)^p\big) < \infty$

Review of Singular Values

- A compact, then $A = \sum_0^\infty s_j \left(e_j \otimes f_j\right)$ with $s_j = \lambda(A^* A)^{\frac{1}{2}}$ the **singular values** with $s_0 \geq s_1 \geq \to 0$
- $s_0 = \|A\|_{H^1 \to H^2}$
- **Proposition B.15** for compact operators A, B then
- $s_{j+k}(A + B) \leq s_j(A) + s_k(B)$
- $s_{j+k}(AB) \leq s_j(A) s_k(B)$
- If A is compact and B is bounded, then $s_j(AB), s_j(BA) \leq \|B\| s_j(A)$

prove $\sum_j s_j\left((VR_0(\lambda)\rho)^p\right) < \infty$

- Let $\rho_1 \in C_c^\infty(R^n)$ with $supp\ \rho_1 \subset B(0,R)$, then we can consider the map
- $\rho_1 R_0(\lambda)\rho_1 : L^2(T_R^n) \to L^2(T_R^n)$ with $T_R = \mathbb{R}/R\mathbb{Z}^n$
- $s_j(\rho_1 R_0(\lambda)\rho_1) = s_j\left(\left(-\Delta_{T_R^n} + 1\right)^{-1}\left(-\Delta_{T_R^n} + 1\right)^1 \rho_1 R_0(\lambda)\rho_1\right)$
- By **proposition B.15** if A is compact and B bounded, then
- $s_j(AB) \leq \|B\| s_j(A)$
- $s_j\left(\left(-\Delta_{T_R^n} + 1\right)^{-1}\left(-\Delta_{T_R^n} + 1\right)^1 \rho_1 R_0(\lambda)\rho_1\right) \leq s_j\left(\left(-\Delta_{T_R^n} + 1\right)^{-1}\right)\left\|\left(-\Delta_{T_R^n} + 1\right)^1 \rho_1 R_0(\lambda)\rho_1\right\|$
- By B.3.9, we get the first term is bounded by $Cj^{-\frac{2}{n}}$
- The second term can be bounded using **theorem 3.1**
- $\|\rho R_0(\lambda)\rho\|_{L^2 \to H^2} \leq C\left\|(1+|\lambda|)^1 e^{L(Im\lambda)_-}\right\|$
- Putting this together, we get
- $s_j(\rho_1 R_0(\lambda)\rho_1) \leq C|\lambda| j^{-\frac{2}{n}} e^{L(Im\,\lambda)_-}$
- Finally, take $\rho_1 = 1$ on $supp\ \rho \cup supp\ V$ to get
- $s_j\left((VR_0(\lambda)\rho)^p\right) \leq C_1 |\lambda|^p j^{-\frac{2p}{n}} \exp(C_1(Im\,\lambda)_-)$
  which is summable when $p \geq \frac{n+1}{2}$

proof

- For a trace class operator A, we can define the Fredholm determinant $\det(I - A)$,
- **Prop B.28**: If A is of trace class, then $I - A$ is invertible if and only if $\det(I - A) \neq 0$
- We set $H(\lambda) = \det(I - (VR_0(\lambda)\rho)^{n+1})$
- let $m_H(\lambda)$ be the multiplicity of a zero of $H(\lambda)$
- **Theorem 3.26 (Multiplicity of resonance)**
- Given H as above and $m_H(\lambda)$ the multiplicity of λ as a zero of $H(\lambda)$, then
- $m_R(\lambda) \leq m_H(\lambda)$ for $\lambda \in \mathbb{C}$

$m_R(\lambda) \leq m_H(\lambda)$

- enough to prove when $m_R(\lambda) \leq 1$ (by theorem 3.14)
- because n is odd, we know that
- $I - (VR_0(\lambda)\rho)^{n+1} = \sum_{j=0}^{n}(-VR_0(\lambda)\rho)^j(I + VR_0(\lambda)\rho)$
- If λ is a simple pole of $R_V(\lambda)$ then since
- $R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0\big(\lambda\,(1-\rho)\big)$
- (3.2.1) we must have that $(I + VR_0(\lambda)\rho)$ has a nontrivial kernel (Fredholm Operator stuff).
- Therefore the LHS has nontrivial kernel therefore $H(\lambda) = 0$ so thet $m_H(\lambda) \geq 1 = m_R(\lambda)$

Relate $m_H(\lambda)$ with growth of H

- **Jensen's formula:**
- If H is holomorphic and $n(t)$ is the number of zero of H in $|z| < t$ then

$$\int_0^t \frac{n(t)}{t} dt + \log|H(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|H\left(e^{(i\theta)} r\right)\right| d\theta$$

- $n(r) \leq \frac{1}{\log 2} \int_r^{2r} \frac{n(t)}{t} dt \leq \frac{1}{\log 2} \left(\log \max_{|z|=2r} |H(z)| - \log|H(0)|\right)$
- We want to now prove $|H(\lambda)| \leq A \exp(A|\lambda|^n)$

bound $H(\lambda)$

- Use Weyl inequality B.5.11
- $|\det(I - A)| \leq \prod_{j=0}^{\infty}\left(1 + s_j(A)\right)$
- where in our case $A = (VR_0(\lambda)\rho)^{n+1}$
- Then we can use proposition B.15 again to get
- $s_j\left((VR_0(\lambda)\rho)^{n+1}\right) = s_j\left((V\rho R_0(\lambda)\rho)^{n+1}\right)$
- $\leq \|V\|_{\infty}^{n+1} s_j((\rho R_0(\lambda)\rho)^{n+1})$
- $\leq \|V\|_{\infty}^{n+1}\left(s_{\left\lfloor\frac{j}{n+1}\right\rfloor}(\rho R_0(\lambda)\rho)\right)^{n+1}$

Bound $H(\lambda)$ for $Im(\lambda) \geq 0$

- So we are after bounding $s_j(\rho R_0(\lambda)\rho)$
- Let's first suppose that $Im\ \lambda \geq 0$, recall that we already proved that
- $s_j(\rho_1 R_0(\lambda)\rho_1) \leq C \min\left(|\lambda|^{-1}, j^{-\frac{1}{n}}, |\lambda| j^{-\frac{2}{n}}\right) \exp(C(Im\ \lambda)_-) \leq C j^{-\frac{1}{n}}$
- $s_k\left((VR_0(\lambda)\rho)^{n+1}\right) \leq \|V\|_\infty^{n+1} C k^{-\frac{n+1}{n}} \leq C_1 k^{-\frac{(n+1)}{n}}$
- $|H(\lambda)| \leq \prod_{k=1}^{\infty}\left(1 + s_k\left((VR_0(\lambda)\rho)^{n+1}\right)\right)$
- Then since $\prod(1 + x_k) \leq \exp \sum x_k$ we get:
- $H(\lambda) \leq \exp\left(C_1 \sum_{k=1}^{\infty} k^{-\frac{n+1}{n}}\right)$

Bound $H(\lambda)$ for $Im(\lambda) < 0$

- Now let $Im(\lambda) < 0$, then by 3.1.19 (Stone's formula for the real laplacian)
- $R_0(\lambda, x, y) - R_0(-\lambda, x, y) = \frac{i\lambda^{n-2}}{2(2\pi)^{n-1}} \int_{S^{n-1}} e^{i\lambda\langle\omega, x-y\rangle} d\omega$
- We can rewrite this as
- $\rho\big(R_0(\lambda) - R_0(-\lambda)\big)\rho = a_n \lambda^{n-2} E_\rho\big(\bar{\lambda}\big)^* E_\rho(\lambda)$
- where $E_\rho(\lambda)u(\omega) := \int_{R^n} e^{i\lambda\langle\omega, x\rangle}\rho(x)u(x)dx \;\; \big(\mathrm{L}^2(\mathrm{R}^\mathrm{n}) \to L^2(S^{n-1})\big)$
- Then we can again use B.3.5 to get
- $s_j(\rho R_0(\lambda)\rho) = s_j\left(a_n \lambda^{n-2} E_\rho\big(\bar{\lambda}\big)^* E_\rho(\lambda) + \rho R_0(-\lambda)\rho\right)$
- Prop B.15 allows us to break up this as:
- $s_{\left[\frac{j}{2}\right]}\left(a_n \lambda^{n-2} E_\rho\big(\bar{\lambda}\big)^* E_\rho(\lambda)\right) + s_{\left[\frac{j}{2}\right]}(\rho R_0(-\lambda)\rho)$
- $\leq a_n |\lambda|^{n-2} \left\|E_\rho\big(\bar{\lambda}\big)\right\| s_{\left[\frac{j}{2}\right]}\left(E_\rho(\lambda)\right) + s_{\left[\frac{j}{2}\right]}(\rho R_0(-\lambda)\rho)$
- $\leq C \exp(C|\lambda|) s_{\left[\frac{j}{2}\right]}\left(E_\rho(\lambda)\right) + Cj^{-\frac{1}{n}}$

Bound $H(\lambda)$ for $Im(\lambda) < 0$

- Now we gotta estimate $s_j\left(E_\rho(\lambda)\right)$, for which we use the Laplacian on a sphere $-\Delta_{S^{n-1}}$
- $s_j\left(E_\rho(\lambda)\right) \leq s_j\left((-\Delta_{S^{n-1}} + 1)^{-l}\right)\left\|(-\Delta_{S^{n-1}} + 1)^l E_\rho(\lambda)\right\|$
- By the Weyl law, we can bound the first term by $C^l j^{-\frac{2l}{n-1}}$
- The second term can be bounded by using the fact that $supp\ \rho \subset B(0,R)$
- $\left\|(-\Delta_{S^{n-1}} + 1)^l E_\rho(\lambda)\right\| \leq C_\rho \sup_{\substack{\omega\in S^{n-1},\\ |x|\leq R}} |(-\Delta_\omega + 1)^l e^{i\lambda\langle x,\omega\rangle}|$
- Then we can bound this by Cauchy estimates, which say
- $\left|f^{(n)}(z_0)\right| \leq \frac{n!}{r^n} M_r$ where M_r is the max value of f on a boundary of a ball $B_r(z_0)$
- so playing around we get: $\left\|(-\Delta_{S^{n-1}} + 1)^l E_\rho(\lambda)\right\| \leq C_1^l \exp(C_1|\lambda|)(2l)!$

Bound $H(\lambda)$ for $Im(\lambda) < 0$

- Then we get $s_j\left(E_\rho(\lambda)\right) \leq C_1^l j^{-\frac{2l}{n-1}} \exp(C_1|\lambda|)(2l)!$
- let's choose a good l, using $(2l)! \leq (2l)^{2l}$ we have:
- $C_1^l j^{-\frac{2l}{(n-1)}}(2l)! \leq \left(\frac{j}{C_3 l^{n-1}}\right)^{-\frac{2l}{n-1}} = \exp\left(-\frac{j^{\frac{1}{n-1}}}{C_4}\right)$
  where $l = \left(\frac{j}{C_3 e}\right)^{\frac{1}{n-1}}$
- Therefore $s_j\left(E_\rho(\lambda)\right) \leq C_3 \exp\left(C_2|\lambda| - \frac{j^{\frac{1}{n-1}}}{C_2}\right)$

finish proof

- Now we can just put everything together, we have:
- $s_k\left((VR_0(\lambda)\rho)^{n+1}\right) \le C\left(s_{\left[\frac{k}{n+1}\right]}(\rho R_0(\lambda)\rho)\right)^{n+1} \le C\left(\exp(C|\lambda|)s_{\left[\frac{j}{2}\right]}(E_\rho(\lambda) + Cj^{-\frac{1}{n}}\right)^{\mathrm{n+1}}$
- $\le C\exp\left(C|\lambda| - \frac{k^{\frac{1}{n-1}}}{C}\right) + Ck^{-\frac{n+1}{n}}$
- So $s_k\left((VR_0(\lambda)\rho)^{n+1}\right) \le \begin{cases} C_4\exp(C_4|\lambda|), & k \le C_4|\lambda|^{n-1} \\ C_4 k^{\frac{-n+1}{n}}, & k \ge C_4|\lambda|^{n-1} \end{cases}$
- Therefore we get:
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- $|H(\lambda)| \le \prod\left(1 + s_k\left((VR_0(\lambda)\rho)^{n+1}\right)\right) \le \exp(C_4|\lambda|)\left(\exp\sum_{k\ge C_4|\lambda|^{n-1}} C_4 k^{-\frac{n+1}{n}}\right) \le \exp(C_5|\lambda|^n)$

Chapter 3.5
Complex Valued Potentials with no resonance

Motivation

- **Theorem 2.16**:
- $\sum\{m_R(\lambda): |\lambda| \leq r\} = \frac{2|chsuppV|}{\pi} r\big(1 + o(1)\big)$ as $r \to \infty$
- so we have infinitely many resonances in dimension 1, this is not the case in higher dimensions.
- There exists potentials with no resonances.

Theorem 3.29 (Complex Valued potentials with no resonances)

- Let $(r, \theta, x^{'})$ be cylindrical coordinates in R^{k+2} with k odd:
- $x = (x_1, x_2, x^{'}), x_1 = r\cos\theta, x_2 = r\sin\theta, x^{'} \in R^k$
- Suppose that $V \in L^{\infty}_{comp}(R^{k+2}, C)$ is of the form:
- $V(x) = e^{i\theta m} W(r, x^{'})$ with $W \in L^{\infty}_{comp}([0, \infty) \times R^k)$
- if $m \neq 0$ then $R_v(\lambda)$ is entire (ie there are no poles and therefore no resonances for $-\Delta + V$)

Proof Outline

- Assume there is a pole, get an L^2 resonant state, compute the norms of the orthogonal projections onto Fourier modes to see they are all zero, and so our resonant state is zero, which gives a contradiction!

resonant state

- Assume $R_V(\lambda)$ has a pole.
- by 3.2.18, we know that
- $R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}\big(I - VR_0(\lambda)(1-\rho)\big)$
- so simplicity of a pole of $R_V(\lambda)$ is equivalent to simplicity of a pole of $(I + VR_0(\lambda)\rho)^{-1}$
- And $m_R(\lambda) > 0$ will imply that $(I + VR_0(\lambda)\rho)^{-1}$ has some pole for any $\rho \in C_c^\infty\big(R^{2+k}\big)$ such that $\rho = 1$ on $supp\, V$.
- Let's take $\rho = \rho(r, x')$
- By results about poles, we know that there exists a $u \in L^2$ such that
- $(I + VR_0(\lambda)\rho)u = 0$
- so $u = -VR_0(\lambda)\rho u = -V\rho R_0(\lambda)\rho u$

Fourier Mode Projections

- Let $\Pi_l u(r,\theta,x^{'}) \coloneqq e^{il\theta}\frac{1}{2\pi}\int_0^{2\pi} u(r,\phi,x^{'})e^{-il\phi}d\phi$ (this is the projection onto the l^{th} fourier mode)
- ρ doesn't depend on θ, therefore Π_l commutes with $\rho R_0(\lambda)\rho$
- since $u = -V\rho R_0(\lambda)\rho u = -e^{im\theta}W(r,x^{'})\rho R_0(\lambda)\rho u$
- therefore $\Pi_{j+m}u = \Pi_{j+m}\left(e^{im\theta}W\rho R_0(\lambda)\rho u\right) = e^{im\theta}W\rho R_0(\lambda)\rho\Pi_j u$
- $\left\|\Pi_{j+m}u\right\|_{L^2} \leq C\left\|\rho R_0(\lambda)\rho\Pi_j u\right\|_{L^2}$

Fourier Mode Bound (Lemma 3.30)

- **Lemma:** $\|\rho R_0(\lambda)\rho\Pi_l f\|_{L^2} \leq \frac{Ce^{C(Im\,\lambda)_-}}{\langle l\rangle}\|f\|_{L^2}$ (for $f \in L^2$)
- proof: let $u \coloneqq \rho R_0(\lambda)\rho\Pi_l f$
- Then by Theorem 3.1 (free resolvent in odd dimensions):
- $\|u\|_{H^1} = \|\rho R_0(\lambda)\rho\Pi_l f\|_{H^1} \leq C\, e^{C\,(Im\,\lambda)_-}\|f\|_{L^2}$
- $\|u\|_{H^1}^2 \geq \langle -\Delta u, u\rangle$

compute $\langle -\Delta u, u\rangle$

- In polar coordinates we have $\Delta = \partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2} + \Delta_{x'} = -D_r^2 + \frac{i\partial_r}{r} + \Delta_{x'} - \frac{D_\theta^2}{r^2}$
- then $-\Delta u = \left(D_r^2 - \left(\frac{i}{r}\right) D_r - \Delta_x' + \frac{l^2}{r^2}\right) u$
- $\langle -\Delta u, u\rangle = \int_{R^k} \int_0^\infty \int_0^{2\pi} \left(D_r^2 - \left(\frac{i}{r}\right) D_r - \Delta_{x'} + l^2/r^2\right) u\bar{u}\, d\theta\, r\, dr\, dx'$
- due to compact support of u we can integrate by parts to get
- $\int_{R^k} \int_0^\infty \int_0^{2\pi} \left(|\partial_r u|^2 + |\nabla u|^2 + \frac{l^2}{r^2} |u|^2\right) d\theta\, r\, dr\, dx'$
- $\geq \left\langle \left(\frac{l^2}{r^2}\right) u, u \right\rangle_{L^2} \geq \frac{l^2 \|u\|_{L^2}^2}{C}$

Fourier Mode Bound

- $\|u\|_{H^1} = \|\rho R_0(\lambda)\rho\Pi_l f\|_{H^1} \leq C\ e^{C\ (Im\ \lambda)_-} \|f\|_{L^2}$
- $\|u\|_{H^1}^2 \geq \langle -\Delta u, u\rangle = \frac{l^2\|u\|_{L^2}^2}{C}$
- $\|u\|_{L^2}^1 \leq C\,\frac{e^{C\ (Im\ \lambda)_-}\|f\|_{L^2}}{|l|} \leq C\,\frac{e^{C\ (Im\ \lambda)_-}\|f\|_{L^2}}{\langle l\rangle}$

Returning

- $\left\|\Pi_{j+m}u\right\|_{L^2} \leq C\left\|\rho R_0(\lambda)\rho\Pi_j u\right\|_{L^2}$
- $C\left\|\rho R_0(\lambda)\rho\Pi_j\Pi_j u\right\|_{L^2} \leq \frac{Ce^{C(Im\ \lambda)_-}}{\langle j\rangle}\left\|\Pi_j u\right\|_{L^2}$
- Let $a_j = \left\|\Pi_j u\right\|_{L^2}$ and $C_j = \frac{Ce^{C|\lambda|}}{\langle j\rangle}$
- **lemma 3.31 (Two sided sequences):**
- Given $\left\{a_j\right\}_{\infty}^{\infty}$ a sequence going to zero in both directions such that for some $m \in Z_{\neq 0}, J \in N$ we have that for all j there exist $C_j \geq 0$ such that $\left|a_{j+m}\right| \leq C_j$ and $C_j \leq 1$ for $|j| \geq J$
- Then $a_j \equiv 0$ for $j \in Z$

proof of two sided sequence

- hypoth: $a_j \to^{|j|\to\infty} 0$, $|a_{j+m}| \le C_j$, $C_{|j|\ge J} \le 1$
- fix j, have
- $|a_j| \le C_{j-m}\,|a_{j-m}| \le \cdots \le \prod_{k=1}^{p} C_{j-km} |a_{j-mp}| \le K |a_{j-mp}| \to 0$
- as $p \to \infty$ where $K \coloneqq \prod_{|l|<J} C_l \ge \prod_{|j-mk|<J} C_{j-km}$
- so $a_j = 0$